Last time:

La 1: Y top. space TFAE:

1) Y compact, Hausdorff & totally disconnected

2) $Y \xrightarrow{\sim} \varprojlim_{I} Y_i$, Y_i finite, discrete

I small cofiltered cat.

3) Y compact, Hausdorff & each $y \in Y$ has basis of compact + open nbhds

Such Y are called profinite sets.

Ex: •) $\mathbb{Z}_p = \varprojlim_{n} \mathbb{Z}/p^n$

•) $\prod_{\mathbb{N}} M$, M finite, discrete

•) Cantorset, $\{0\} \cup \{\frac{1}{n} \mid n \geq 1\}$

Prf: 2) $\Rightarrow$ 3)

$$Y = \varprojlim_{I} Y_i = \Big\{ (y_i)_i \in \prod_I Y_i \;\Big|\; \begin{array}{l} \forall\, j \to i \text{ in } I \text{ with} \\ \text{ass. } g_{ij} : Y_j \to Y_i \\ \text{we have} \\ g_{ij}(y_j) = y_i \end{array} \Big\}$$

$$\subseteq \prod_I Y_i \quad \leftarrow \text{compact \& Hausdorff}$$

closed subspace

Let $\pi_i : Y \to Y_i$ be the can. proj.,

$y \in Y$

By def of top on Y + Y_i discrete

$U_i := \pi_i^{-1}(\pi_i(y))$ bas. of closed & open nbhds of y

$\ni$
y

closed in Y $\Rightarrow$ compact

3) $\Rightarrow$ 1): $x, y \in Y, x \neq y$

$\Rightarrow$ $\exists\, U, W \subseteq Y$ cpct, open

s.t. $x \in U$, $y \in W$

& $x \notin W$, $y \notin U$

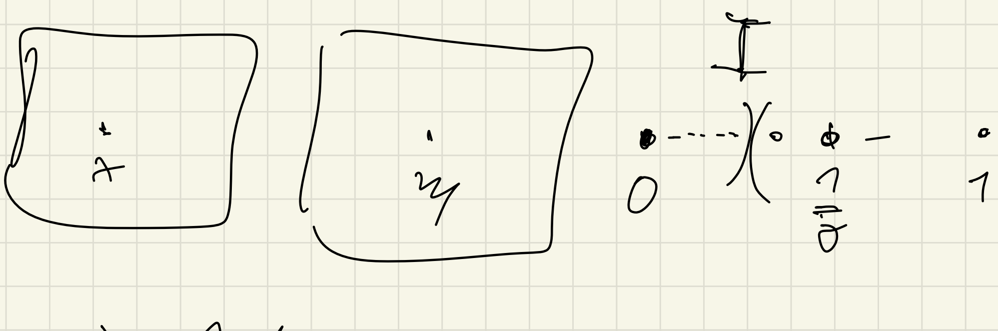
$\Rightarrow$ $U' := U \setminus W$, $W' := W \setminus U$

$\Rightarrow$ $x \in U'$, $y \in W'$,

U', W' cpct, open

& $U' \cap W' = \emptyset$

Def. of tot. disc.



1) $\Rightarrow$ 2): Set

$I := \{ Y \xrightarrow{f} F \mid f$ quotient map, F finite, discrete$\}_{/\cong}$

For $i \in I_f$ write $Y \xrightarrow{f_i} Y_i$
$= (Y \xrightarrow{f} Q)$
for corresp. surj. $\quad f_i = f, \; Y_i = Q$

Define $i \geq j$ if ex. fact.

$$\begin{tikzcd}
Y \arrow[r, two heads, "f_j"] \arrow[dr, two heads, "f_i"'] & Y_j \arrow[d, dashed, "\exists ?\, g_{ij}"] \\
& Y_i
\end{tikzcd}$$

nec. unique if exist

and $i \simeq j$ if $i \leq j$ & $j \leq i$

$\Rightarrow I$ partially ordered set, cofiltered (take products

$Y \to Y_i \times Y_j \leftarrow$ finite, discrete)
$\twoheadrightarrow Y_k$

Set $Z := \varprojlim_{I} Y_i$

$\Rightarrow$ get can. cont. morph.
$Y \xrightarrow{\varphi} Z$

Y, Z cpct, Hausdorff (by 2) $\Rightarrow$ 1))

$\Rightarrow$ STP φ bijective

i) φ injective

Pick $x, y \in Y$, $x \neq y$

$\exists$ U, W open, cld, $x \in U$, $y \in W$,

$U \cap W = \emptyset$

Define $\Upsilon = \{0,1\} \xleftarrow{f} Y$

by $f_{|U} \equiv 0$, $f_{|Y \setminus U} \equiv 1$

$\Rightarrow$ f cont. + quotient map

$f(x) \neq f(y) \Rightarrow \varphi(x) \neq \varphi(y)$

ii) φ surj.

Let $z \in Z$, $\pi_j : Z \to Y_j$ can. proj.

Set $U_j := \pi_j^{-1}(\pi_j(Z))$ closed in Y (+open)

$\Rightarrow \varphi^{-1}(Z) = \bigcap_j U_j$ by def. Z

Note: $j \leq i \Rightarrow U_j \subseteq U_i$ $(g_{ij}(z_j) = z_i)$

Assume $\bigcap_j U_j = \emptyset$

$\Rightarrow Y = \bigcup_j \underbrace{Y \setminus U_j}_{\text{open}}$

$\Rightarrow Y = Y \setminus U_j$ for some j

Y cpct

$\Rightarrow U_j = \emptyset$ $\lightning$

$\left(\text{as } \begin{tikzcd} Y \arrow[r] \arrow[rr, bend right, two heads] & Z \arrow[r, "\pi_j"] & Y_j \end{tikzcd} \right)$

/Prop. $\square$

Def: G top. grps TFAE:

⚠ 1) $G \cong \varprojlim_I G_i$, G_i finite, discrete

$G \to G_i$ I cofiltered cat

$\downarrow$ 2) $1 \in G$ has basis of nbhds of open, compact, normal subgroups U_i

$\ker(G \to G_i) = U_i$

& G compact, Hausdorff

3) The underlying top. space of G is a profinite set

Such G are called profinite groups

Pf: 1) $\Leftrightarrow$ 2) similar to before

2) $\Leftrightarrow$ 3) SP, Tag 0BR1

Ex: 1) $\mathbb{Z}_p$ (even profinite ring)

$$\mathbb{Z}_p \supset \mathbb{Z}_p^* = \varprojlim_n (\mathbb{Z}/p^n)^*$$

2) H any group

$$\hat{H} = \varprojlim_{\substack{N \trianglelefteq H \\ \text{normal} \\ \text{of finite} \\ \text{index}}} H/N$$ "profinite completion"

e.g. $\widehat{\mathbb{Z}_{(p)}} = \mathbb{Z}_p$

CRT

$$\hat{\mathbb{Z}} = \varprojlim_{n} \mathbb{Z}/n \underset{\text{as top. rings}}{\cong} \prod_p \mathbb{Z}_p$$

3) (Exercise) L/K normal + separable

$\Rightarrow$ $Gal(L/K) := \{\sigma : L \xrightarrow{\sim} L \mid \sigma_{|K} = Id_K\}$

profinite (for compact-open top.)

$$\cong \varprojlim_{\substack{K \subseteq M \subseteq L \\ [M:K] \\ M/K \text{ Galois}}} Gal(M/K)$$

$\{ \text{subfields } M \subseteq L/K \}$

$\xleftrightarrow{1:1} \{ \underline{\text{closed}} \text{ subgroups } H \subseteq Gal(L/K) \}$

Particular important: $L = K^{sep}$

$\Rightarrow Gal(K^{sep}/K)$ "absolute Galois group of K"

$\mathbb{Q}_p$ as a completion of $\mathbb{Q}$

Fix prime p

Define $v_p : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$

$$x \mapsto \begin{cases} \infty & , x = 0 \\ a & , \text{if } x = p^a \frac{m}{n}. \end{cases}$$

"p-adic valuation" $\quad m, n \in \mathbb{Z}, p \nmid m \cdot n$

& $|\cdot| = |\cdot|_p : \mathbb{Q} \to \mathbb{R}_{\geq 0}, x \mapsto \left(\frac{1}{p}\right)^{v_p(x)}$
"p-adic norm"

Then $|\cdot|_p$ is a norm on $\mathbb{Q}$, i.e.

1) $|x| = 0 \Leftrightarrow x = 0$

2) $|x \cdot y| = |x| \cdot |y|$

3) $|x+y| \leq |x| + |y|$

(like usual absolute value $|\cdot|_\infty : \mathbb{Q} \hookrightarrow \mathbb{R} \xrightarrow{|\cdot|_\infty} \mathbb{R}_{\geq 0}$)

Indeed,

1) ✓ ($\Leftarrow\Rightarrow$ ($v_p(x) = \infty \Leftrightarrow x = 0$))

2) translates into

$$v_p(xy) = v_p(x) + v_p(y)$$

3) $x = p^{v_p(x)} \frac{m}{n}$, $y = p^{v_p(y)} \cdot \frac{c}{d}$

$p \nmid m \cdot n \cdot c \cdot d$

$$\Rightarrow x + y = p^{\min(v_p(x), v_p(y))} \cdot \frac{e}{f},$$

$p \nmid f$ (but $p \mid e$ possible)

$$\Rightarrow v_p(x+y) \geq \min(v_p(x), v_p(y))$$

$\Leftrightarrow \quad |x+y| \leq \max(|x|, |y|) \; (\leq |x|+|y|)$

"ultrametric triangle inequality"

Def: A valued field $(K, |\cdot|)$ is a field K together with an abs. value (or norm) $|\cdot|: K \to \mathbb{R}_{\geq 0}$, s.t. 1), 2), 3) from above hold.

Ex: * $(\mathbb{Q}, |\cdot|_p)$, $(\mathbb{Q}, |\cdot|_\infty)$
$(\mathbb{R}, |\cdot|_\infty)$, $(\mathbb{C}, |\cdot|_{usual})$

* K any field,

$$|\cdot| : K \to \mathbb{R}_{\geq 0}$$
$$x \mapsto \begin{cases} 0, & x = 0 \\ 1, & x \neq 1 \end{cases}$$

"trivial absolute value"

Note: $(K, |\cdot|)$ valued field

$\Rightarrow$ $K \times K \to \mathbb{R}_{\geq 0}$, $(x,y) \mapsto |x-y|$

defines a metric on K & thus a top. on K. For this topology K is a a topological field, i.e. add/mult/inv. are cont.

E.g.: $(\mathbb{Q}, |\cdot|_p)$

$\Rightarrow$ $U_a = \{ x \in \mathbb{Q} \mid |x|_p \leq p^{-a} \}$ for $a \in \mathbb{Z}$

open, $U_a = \{ x \in \mathbb{Q} \mid |x|_p < p^{-a+1} \}$

& $\{U_a\}_a$ form a basis of nbds of 0

In part,

$$U_0 = \mathbb{Z}_{(p)} = \left\{ \frac{m}{n} \in \mathbb{Q} \;\middle|\; p \nmid n,\ m,n \in \mathbb{Z},\ n \neq 0 \right\}$$

is open & the subspace top. on $\mathbb{Z}_{(p)}$ is the (p)-adic top.

(as $U_a = p^a \cdot \mathbb{Z}_{(p)}$ for $a \geq 0$)

1) $R = \mathbb{Z}_{(p)}$, $I \subseteq R$, $I = (p)$, $\hat{R}_I = \mathbb{Z}_p$

2) $R = \mathbb{Q}$, $I = (p)_{\mathbb{Q}} \Rightarrow \hat{R}_I = 0$

$$(p)_{\mathbb{Q}} = (1)_{\mathbb{Q}}$$

Prop 3: $(K, |\cdot|)$ valued field. There exists a unique (up to unique isom) valued $(\hat{K}, |\cdot|_{\hat{K}})$, s.t.

1) $K \hookrightarrow \hat{K}$, $|\cdot|_{\hat{K}}$ restricts to $|\cdot|$

2) K is dense in $\hat{K}$

3) $\hat{K}$ is complete, i.e. Cauchy seq. converge

4) If $(K, |\cdot|) \hookrightarrow (L, |\cdot|)$ is an embedding of valued fields, L complete, then there exist unique ext. $(\hat{K}, |\cdot|) \hookrightarrow (L, |\cdot|)$

E.g.: 1) The completion of $(\mathbb{Q}, |\cdot|_\infty)$ is $(\mathbb{R}, |\cdot|_\infty)$

2) $(\mathbb{Q}, |\cdot|_p)$ is $(\mathbb{Q}_p, |\cdot|_p)$

Prf of Prop 3: Prop. 8.2.2. in Tian
(set $\hat{K}$ as equiv. classes of Cauchy seq. with val. in K) $\square$

Def: $(K, |-|)$ valued field
1) $|-|$ non-archimedean if
$|x+y| \leq \max(|x|, |y|) \; \forall x, y \in K$
2) $|-|$ archimedean if $|-|$ is not non-archimedean
3) $|-|_1, |-|_2$ norms on K are equivalent if $|-|_2 = |-|_1^r$ for some $r > 0$

Note: If $|-|_1, |-|_2$ are equivalent, they define the same top. on K

Let's analyze non-arch. valued fields further.

Assume $(K, |\cdot|)$ non-arch valued field, fix $q > 1$ real number

set
$$v(x) := -\log_q(|x|)$$
$$\rightsquigarrow v : K \to \mathbb{R} \cup \{\infty\}$$

Then

1) $v(x) = \infty \iff x = 0$
2) $v(x \cdot y) = v(x) + v(y)$
3) $v(x+y) \geq \min(v(x), v(y))$

"v is a (n additive) valuation"

Conversely, given any add. val.
$$v : K \to \mathbb{R} \cup \{\infty\}$$

$|x| := q^{-v(x)}$ defines a n.a. norm

on K, different choices of q give equivalent norms, equivalent norms define equivalent (add.) valuations in the sense $v_1 \sim v_2$ if ex.

$r > 0$, s.t. $v_1 = r \cdot v_2$

$\uparrow$

$\mathbb{R}$